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On a Property of Mr. Gompertz's Law of Mortality. By PROFESSOR DE MORGAN.

IT is commonly known that in calculating an annuity on three joint lives, two of the lives are made to count as one life of the same annuity value; and that the result is approximately true. Twenty years ago (*Phil. Mag.*, Nov., 1839), I showed that if the law of mortality explained by Mr. Gompertz were accurately true for the whole of life, this substitution of one life instead of two would also be accurately true. I now repeat the same proposition in a more general form.

Required the law of mortality under which the table of two lives follows the same law as the table of one life—that is, in which the chance of two or more lives surviving a given term is always equal to the chance of some one life, older than either, surviving a term of the same length. Had any one, observing the usual rule for determining an annuity on three lives, had the curiosity to inquire what is the law of mortality for which it is accurately true, he would have arrived at Mr. Gompertz's law of mortality by a totally different route. I do not think it right to occupy space by a very full development of the demonstration: the following will be enough for anyone who has an ordinary acquaintance with functional algebra and the differential calculus.

Let the number living at the age x be $\epsilon^{\phi x}$, ϕx being an un-

known function of x. Let the chance of lives aged x and y both surviving n years be equal to that of a single life of the age z surviving n years—z being an unknown function of x and y, but not of n. We have then,

$$\epsilon^{\phi(x+n)-\phi x} \times \epsilon^{\phi(y+n)-\phi y} = \epsilon^{\phi(z+n)-\phi z};$$

$$\text{or, } \phi(x+n)+\phi(y+n)-\phi x-\phi y=\phi(z+n)-\phi z.$$

This being true for all values of n, and x, y, z, being independent of n, differentiation with respect to n gives

$$\phi'(x+n)+\phi'(y+n)=\phi'(z+n).$$

Now this equation is not possible for all values of x, y, n, except under the condition that $\phi'(x+n)$ is of the form $fx.\mathrm{F}n$; and even this form is limited to $fx.fn$, since x and n must be transposable without alteration of value in $\phi'(x+n)$. Hence $\phi'x$ is $fx.f(0)$, whence fx is of the form $a\phi'x$; a being independent of x. Hence $\phi'(x+n)$ is $a^2\phi'x.\phi'n$, and $a^2\phi'(x+n)$ is always $a^2\phi'x.a^2\phi'n$, which it is known cannot be unless $a^2\phi'x$ be of the form ϵ^{cx}, whence ϕx is of the form $b\epsilon^{cx}+k$; b, c, k, being constants independent of x. Hence

$$\epsilon^{\phi x}=n^{\circ} \text{ living at age } x=\epsilon^{k+b\epsilon^{cx}}=dg^{q^x},$$

to use Mr. Gompertz's letters; here $d=\epsilon^k$, $g=\epsilon^b$, $q=\epsilon^c$.

Mr. Gompertz's paper "On the Nature of the Function expressive of the Law of Human Mortality," read before the Royal Society, June 16, 1825, is not by any means so well known as it ought to be, even by actuaries. It is the first attempt at an hypothesis on the law of mortality from a physiological point of view. If of 10,000 persons of a given age, one may be expected to die in a week, while of 10,000 persons of a given older age two may be expected to die in a week, it is clear that the power of death over the second set is double of the same power over the first set. Another way of stating it is, that the power of the second set to oppose destruction is half of the power of the first set. Either phrase may be adopted, and either may be objected to; all that is meant is, that the fight between life and death which is always going on gives death twice as large a proportion of his possible victims in the second case as in the first. The *intensity of mortality* is twice as great in the second case as in the first. If l_x represent the number living at the age x, and if $fx.dx$ represent the *proportion* who die in the next interval dx, then fx may represent the intensity of mortality at the age x; and we have $dl_x=-l_x fx.dx$, the negative sign being required to express that death diminishes the number of the living. The physiological law

assumed by Mr. Gompertz is, that the intensity of mortality gains equal proportions in equal times; so that in whatever ratio it is augmented in any given interval, it is augmented in the same ratio in any other interval of the same length. That is, $fx = aq^x$; a and q being independent of x. Hence

$$dl_x = -l_x aq^x dx, \quad \text{log. } l_x = \text{const.} - \frac{aq^x}{\text{log. } q},$$

whence the form $l_x = \text{A}.g^{q^x}$ is easily obtained; g being some quantity less than unity, and $\text{log.}\, g \ \text{log.}\, q.q^x$ being the intensity of mortality at the age x. The logarithms are Naperian.

In comparing his theory with tables, Mr. Gompertz found that it applies with singular accuracy through long periods, but requires a change of the constants at one or more periods. In the Carlisle Tables he found—the logarithms being common logarithms—

$$\text{From 10 to 60, log. } l_x = 3{\cdot}88631 - 10^{\bar{2}\cdot75526+\cdot0126x}.$$
$$\text{From 60 to 100, log. } l_x = 3{\cdot}79657 - 10^{\bar{3}\cdot74767+\cdot02706x}.$$

The law itself, when applied, indicates a slow change in the quantities supposed constant. Thus, taking the interval from 10 to 58 years of age, Mr. Gompertz found

$$\text{log. } l_x = 3{\cdot}89063 - 10^{\bar{2}\cdot784336+\cdot0120948x}.$$

To return to the first view taken of this law in the present paper, the value of z, the age of the single life equivalent to the joint lives aged x and y, is to be determined from the equation

$$q^x + q^y = q^z; \quad \text{or, } z = \frac{\text{log. } (q^x + q^y)}{\text{log. } q}.$$

We can now institute comparisons by help of any table for which annuities have been calculated. Equal lives will be the most convenient. If $x = y$, we have

$$z = x + \frac{\text{log. } 2}{\text{log. } q};$$

that is, if Gompertz's hypothesis were true throughout the whole of life, without any change of constants, the seniority of the life equivalent to two equal lives would always be of the same amount for every kind of annuity and every rate of interest. In the case of log. $q = {\cdot}0126$, it would be ·30103 divided by ·0126, or 24 years very nearly. For two lives aged 20, at 3 per cent., the equivalent single life, for an annuity on the whole lives, is close to 37 years, with a seniority of only 17 years. For a life annuity of 10 years, the seniority will be found to be 21 years, and the same for an

annuity of 40 years. The comparisons are so easy that I do not think it necessary to give more examples. This last theorem will materially assist in judging of the closeness of the approximation, and in making a first attempt at the value of q, when the table of facts is given.

It may be suspected that a value of q derived from a considerable average of short annuities would, when combined with properly altered values of A and g, give a table still nearer to the original than that obtained by Mr. Gompertz's constants, though this last is surprisingly near. Very slight changes in the value of q make differences of years in the *seniority*. The effect of the rate of interest is comparatively small. At 6 per cent., with lives both aged 20, and an annuity for the whole life, the seniority is 18 years; at 3 per cent., 17, as above.

The law of uniform seniority, as it may be called, is true for any given and uniform interval of age. If $y=x+h$, we have

$$z=x+\frac{\log.\ (1+q^h)}{\log.\ q}.$$

On the Mortality amongst American Assured Lives. By SAMUEL BROWN, F.S.S., *Actuary of the Guardian Assurance Company.*

[Read before the Institute of Actuaries, 2nd May, 1859, and ordered by the Council to be printed.]

THE extension of life assurance in the United States, and, especially, the more or less successful attempts of English Companies to compete with the American Companies, render an inquiry into the mortality of American assured lives of the greatest interest to the members of this Institute. The question assumes the greater importance, because, from various reasons, the population statistics of the United States have hitherto been, and will for a long period probably continue to be, in a very imperfect state, notwithstanding the talents and skilful labours of the able men to whom the collection of the data in the different censuses has been entrusted, and the readiness of the people to afford the information desired. The marvellous rapidity with which the population has there increased; the vast extent of the country showing such a diversity of soil and climate; the conversion of the country into town districts, or of wild insalubrious localities into lands reclaimed and rendered